Attacking Cryptographic Schemes Based on “Perturbation Polynomials”

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Abstract—We show attacks on several cryptographic schemes that have been proposed for achieving various security goals in sensor networks. Roughly speaking, these schemes all use “perturbation polynomials” to add “noise” to polynomial-based systems that offer information-theoretic security, in an attempt to increase the resilience while maintaining efficiency. We show that the heuristic security arguments given for these modified schemes do not hold, and that they can be completely broken once we allow even a slight extension of the parameters beyond those achieved by the underlying information-theoretic schemes.

Our attacks apply to the key predistribution scheme of Zhang et al. (MobiHoc 2007), the access-control schemes of Subramanian et al. (PerCom 2007), and the authentication schemes of Zhang et al. (INFOCOM 2008).

I. INTRODUCTION

Implementing standard security mechanisms in sensor networks is often challenging due to the constrained nature of sensor nodes: they have limited battery life, relatively low computational power, and limited memory. As such, a significant body of research has focused on the design of special-purpose, highly efficient cryptographic schemes for sensor network applications.

Here, we examine an approach based on “perturbation polynomials” that has been used to construct several recent schemes [7], [5], [6]. This approach, initiated by Zhang, Tran, Zhu, and Cao [7] and Subramanian, Yang, and Zhang [5], takes a polynomial-based scheme that offers information-theoretic (i.e., perfect) security for some “resilience parameter” $t$ — e.g., a bound on the number of compromised nodes or the number of messages authenticated — and then modifies this underlying scheme so that the resilience is supposedly increased against a computationally bounded attacker. The common idea is to add a small amount of “noise” to the low-degree polynomials used in the original scheme; the claim is that the presence of this noise makes breaking the scheme infeasible even in regimes well beyond the original resilience parameter. Unfortunately, we show here that this naive view is unfounded.

We describe efficient attacks against the schemes from [7], [5], [6], demonstrating that these scheme do not offer any better resilience than the original, information-theoretic schemes on which they are based. We provide theoretical justification as to why our attacks work, as well as experimental evidence that convincingly illustrates their effectiveness. Our results cast strong doubt on the viability of the “perturbation polynomials” approach for the design of secure cryptographic schemes.

A. Organization of the Paper

In this paper, we focus the bulk of our attention on the key predistribution scheme of Zhang et al. [7]. A description of their scheme, and details of our attack, are given in Section II. Our attacks against the message authentication schemes of Zhang et al. [6] are described in Section III. Due to lack of space, we do not include the details of our attacks against the secure data storage/retrieval protocol suggested by Subramanian et al. [5].

II. THE KEY PREDISTRIBUTION SCHEME OF ZHANG ET AL.

A. Background

Schemes for key predistribution enable nodes in a large network to agree on pairwise secret keys. Before deployment, a central authority loads some secret information $s_i$ onto each node $i$, for $i \in \{1, \ldots, N\}$ (where $N$ is the network size). Later, any two nodes $i$ and $j$ can agree on a shared key $k_{i,j}$ of length $\kappa$ using their respective secret information. (Probabilistic schemes, where two nodes are only able to compute a shared key with high probability, have also been considered but will not concern us here.) The security goal is to offer resilience as large as possible, where a scheme has resilience $t$ if an adversary who compromises $t$ nodes $I = \{i_1, \ldots, i_t\}$ is still unable to derive any information about the shared key $k_{i,j}$ for any $i, j$ such that $i, j \not\in I$. Efficiency considerations require computation of the shared keys to be fast, thus ruling out standard public-key approaches, and dictate that the storage (i.e., the size of the keying information $s_i$) should be minimized.

One simple approach is for all nodes to share a single key $k$ (i.e., set $s_i = k$ for all $i$) that is used also as the pairwise key for any pair of nodes. While having minimal storage, this scheme has resilience $t = 0$ since it is completely broken after only one node is compromised. A second trivial approach is
for each pair of nodes to store an independent key. This has optimal resilience \( t = N \), but the storage requirement of \( \binom{N}{2} \cdot \kappa \) is unacceptably high.

Blundo et al. [3] show that resilience \( t \) requires storage \((t+1) \cdot \kappa \) if information-theoretic security is desired; Blom [2] and Blundo et al. show schemes meeting this bound. Let \( \mathbb{F} \) be a field whose elements can be used as pairwise keys. To achieve resilience \( t \) using the scheme of Blundo et al., the authority chooses a random symmetric, bivariate polynomial \( F \in \mathbb{F}[x, y] \) of degree \( t \) in each variable as the master secret key; a node with identity \( i \in \mathbb{F} \) is given the univariate polynomial \( s_i(y) = F(i, y) \) as its secret information. The shared key \( k_{i,j} \) between nodes \( i, j \) is \( s_i(j) = F(i, j) = s_j(i) \), which both parties can compute (using the fact that \( F \) is symmetric). It is not hard to see that an attacker who compromises at most \( t \) nodes learns no information about any key that is shared between non-compromised nodes. However, an attacker who compromises \( t + 1 \) nodes can use interpolation to recover the master polynomial and thus recover all the keys in the system.

### B. The Scheme of Zhang et al.

Zhang et al. [7] suggested a “noisy” version of the above scheme, and claimed that the new scheme has improved resilience for some fixed amount of storage. Roughly, their idea is to give node \( i \) a polynomial \( s_i(y) \) that is “close”, but not exactly equal, to \( F(i, y) \). Nodes \( i \) and \( j \) can compute \( s_i(j) \) and \( s_j(i) \) as before; these results will no longer be equal, but because they are close they can still be used to derive a shared key (by, e.g., using the high-order bits). The hope was that the addition of noise to the nodes’ secret information would prevent reconstruction of the master secret \( F \) even if an adversary corrupts many more than \( t+1 \) nodes; in fact, Zhang et al. claim optimal resilience \( t = N \) as long as the adversary is computationally bounded. (Of course, for a computationally unbounded adversary the lower bound from [3] applies.) We show that this is not the case.

We first describe their scheme in further detail. Let \( p \) be prime, and let \( r < p \) be a “noise bound”. Elements in \( \mathbb{Z}_p \) are represented as integers in \([0, p - 1]\) in the natural way; \( a < b \) means that the integer representation of \( a \) is smaller than the integer representation of \( b \). Their scheme operates as follows:

#### Pre-distribution:

The authority chooses a random symmetric, bivariate polynomial \( F \in \mathbb{Z}_p[x, y] \) of degree \( t \) in each variable. It also chooses at random two univariate degree-\( t \) “noise polynomials” \( g(y), h(y) \) over \( \mathbb{Z}_p \). Let Smal be the set of points for which both \( g(y) \) and \( h(y) \) are small; that is:

\[
\text{Small} \overset{\text{def}}{=} \{ y \in \mathbb{Z}_p : g(y), h(y) \in [0, r] \}.
\]

For any fixed \( y \in \mathbb{Z}_p \), the probability (over choice of \( g, h \)) that \( y \in \text{Small} \) is \( r^2/p^2 \). The authority finds (in time \( O(N \cdot p^2/r^2) \)) a set of \( N \) points \( x_1, \ldots, x_N \) in Small. For each node \( i \), the authority chooses a random bit \( b_i \) and gives node \( i \) the point \( x_i \) and the univariate polynomial

\[
s_i(y) = F(x_i, y) + b_i \cdot g(y) + (1 - b_i) \cdot h(y).
\]

Namely, the noise polynomial is chosen as either \( g(y) \) or \( h(y) \), depending on the random bit \( b_i \).

#### Key agreement:

To compute a shared secret key, nodes \( i \) and \( j \) exchange their points \( x_i, x_j \) and then node \( i \) computes \( s_i(x_j) \mod p \) and node \( j \) computes \( s_j(x_i) \mod p \). Since

\[
s_i(x_j), s_j(x_i) \in \{F(x_i, x_j), \ldots, F(x_i, x_j) + r\},
\]

the points computed by the two parties are close enough that they can be used to obtain a shared key by taking, e.g., the high-order bits of their respective results. (Zhang et al. describe an interactive protocol to handle wraparound, but this is irrelevant for the attacks we describe.)

#### Suggested parameters.

The expected size of \( \text{Small} \) (over random choice of \( g, h \)) is \( r^2/p \), so we require \( r^2/p \geq N \). (Zhang et al. suggest a way to guarantee that the size of \( \text{Small} \) is at least \( r^2/p \), but this still requires \( r^2/p \geq N \).) The shared key has length roughly \( \log(p/r) \), but \( p/r \) cannot be too large since the predistribution phase requires \( O(N \cdot p^2/r^2) \) work. If a larger key is desired, multiple instances of the scheme can be run in parallel and the derived pairwise keys concatenated.

In Table I we list the parameters suggested by Zhang et al. We note that with these parameters, each node must store a secret key of size \( \approx 250 \kappa \) bits in order to compute \( \kappa \)-bit shared keys. For the same amount of storage, the original scheme of Blundo et al. would give information-theoretic resilience to compromise of about 250 nodes. In contrast, Zhang et al. claim (computational) resilience \( t = N \), an improvement of 1–2 orders of magnitude. As we will see, this claim is unfounded and the scheme can be broken by an attacker who compromises \( t + 3 \leq 80 \) nodes. Thus the scheme of Zhang et al. is less resilient (as well as less efficient) than the original scheme of Blundo et al.

### C. Warm-Up: A Simple Attack Using Error Correction

We begin by describing a relatively simple attack on the scheme as described above. We discuss two variants: a very efficient attack that requires corruption of \( \approx 4t \) nodes, and an attack that runs in time \( O(r) \) but requires corruption of only \( \approx 3t \) nodes. Both attacks rely on the error-correction algorithm of Ar, Lipton, Rubinfeld, and Sudan [1].

The first attack works as follows: Compromise \( n = 4t + 1 \) nodes with points \( x_1, x_2, \ldots, x_n \) to obtain the \( n \) polynomials

<table>
<thead>
<tr>
<th>modulus ( p )</th>
<th>noise ( r )</th>
<th># of nodes ( N )</th>
<th>degree ( t )</th>
<th>storage per node (per key-bit)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2^{32} - 5</td>
<td>2^{22}</td>
<td>2^{12}</td>
<td>76</td>
<td>246 bits</td>
</tr>
<tr>
<td>2^{36} - 5</td>
<td>2^{24}</td>
<td>2^{12}</td>
<td>77</td>
<td>246 bits</td>
</tr>
<tr>
<td>2^{40} - 87</td>
<td>2^{26}</td>
<td>2^{12}</td>
<td>77</td>
<td>234 bits</td>
</tr>
<tr>
<td>2^{40} - 87</td>
<td>2^{28}</td>
<td>2^{16}</td>
<td>77</td>
<td>273 bits</td>
</tr>
</tbody>
</table>

TABLE I

THE SUGGESTED PARAMETERS FOR THE KEY PREDISTRIBUTION SCHEME OF ZHANG ET AL.
Choose a point $x^* \in \mathbb{Z}_p$ belonging to any non-compromised node $v^*$, and compute $y_i = s_i(x^*)$ for $i = 1, \ldots, n$. By construction of the $s_i$ we have

$$s_i(x^*) = F(x_i, x^*) + b_i \cdot g(x^*) \cdot (1 - b_i) \cdot h(x^*) = f^*(x_i) + \text{noise}_i,$$

where $f^*(x_i) \overset{\text{def}}{=} F(\cdot, x^*), \text{noise}_0 \overset{\text{def}}{=} h(x^*), \text{and } \text{noise}_1 \overset{\text{def}}{=} g(x^*)$.

Define $f^*_i(x) \overset{\text{def}}{=} f^*(x) + \text{noise}_i$. Considering the set of pairs $\{(x_i, y_i) : i = 1, \ldots, n\}$, we have that for all $i$ either $y_i = f^*_i(x_i)$ or $y_i = f^*_i(x_i)$. Hence, for at least one of $b = 0$ or $b = 1$ we have $y_i = f^*_i(x_i)$ for at least $2t + 1$ values of $i$. Applying the error-correction algorithm of Ar et al. [1], we can recover at least one of the polynomials $f^*_0(\cdot)$ or $f^*_1(\cdot)$.

Once we have either of these polynomials, we can compute the shared key between the node $v^*$ (that is associated with the point $x^*$) and any other node in the network: to get the shared key between $v^*$ and another node $v'$ associated with the point $x'$, we compute $y = f^*_i(x_i) = f^*(x_i) + \text{noise}_i$. Since $\text{noise}_i \in [0, r]$, we see that $y$ is close to $f^*(x_i) = F(x_i, x^*)$. Hence the high-order bits of $y$ are (essentially) equal to the shared key between the two nodes.

A variant of the attack, which requires corrupting only $\approx 3t$ nodes, is as follows. Define $f^*_i, f^*_i$, and $\text{noise}_0$ as above; here, set $n = 3t + 1$. Assume without loss of generality that $\text{noise}_1 > \text{noise}_0$, and treat the value $\delta \overset{\text{def}}{=} \text{noise}_1 - \text{noise}_0$ as known. (Enumerating over all possible values of $\delta$ increases the running time by a multiplicative factor of $r$.)

Corrupt $n$ nodes and compute the set $S = \{(x_i, y_i) : i = 1, \ldots, n\}$ as above. Construct $S'$ by adding to $S$ an additional $n$ tuples $\{(x_{n+i}, y_{n+i})\}$ where $x_{n+i} = x_i$ and $y_{n+i} = y_i - \delta$. Observe that for every tuple $(x_i, y_i) \in S'$ it holds that either $f^*_0(x_i) = y_i$, or $f^*_1(x_i) = y_i$. Moreover, for $1 \leq i \leq n$, either $f^*_0(x_i) = y_i$ or $f^*_0(x_i) = y_i + \delta$. For exactly $n$ tuples $(x_i, y_i) \in S'$ it holds that $f^*_0(x_i) = y_i$. The error-correction algorithm of Ar et al. can thus be used to recover $f^*_0$. (The rest of the attack proceeds as before.)

For the parameters in Table I it always holds that $3t + 1 < 233$ and so this already shows that the scheme performs worse than the original, perfectly secure scheme of Blundo et al. In the following section we show that even a generalized version of the scheme that uses more noise (and is not susceptible to the attack described in this section) is vulnerable to attack, and moreover using only $\approx t$ corruptions.

D. A Generalized Scheme

The attack described in the previous section relies strongly on the fact that the same noise polynomial is used half the time. This suggests an easy patch that foils the attack described in the previous section: Let $g, h$, and Small be as in Section II-B, and let $u$ be “small” relative to $p$ (we will see exactly how small below). Now for each node $i$ choose random $\alpha_i, \beta_i \in [-u, u]$, and give to node $i$ (with identity $x_i$) the univariate polynomial

$$s_i(y) = F(x_i, y) + \alpha_i \cdot g(y) + \beta_i \cdot h(y).$$

This generalizes the scheme of Zhang et al., since their scheme can be obtained by setting $\alpha_i = b_i, \beta_i = 1 - b_i$. The “error polynomial” $\alpha_i g(y) + \beta_i h(y)$ still evaluates to a value in a small range (namely, $[-2ur, 2ur]$) on every point in Small, and so this still allows every pair of parties to compute a shared key.

The noise is now larger than in the scheme of Zhang et al. by a factor of $4u$, so for the same values of $p, r, t$ the pairwise keys will have roughly $\log(4u)$ fewer bits. Still one might hope that this modification would make the scheme more secure, even for small values of $u$. Unfortunately, this is not the case, and below we present an attack that breaks also this more general scheme in time (roughly) $O((t^3 + t \cdot (2u)^3))$ using only $t + 3$ compromised nodes. Note that $u = 1$ for the original scheme of Zhang et al., and so this gives a very efficient attack on their scheme (using fewer compromised nodes than the attack of the previous section). Furthermore, $u$ cannot be too large: we need $4ur < p$ in order for even a single-bit shared key to be derived, meaning that in the worst case (for the attacker) the running time of the attack is $O((t^3 + t \cdot (p/2r)^3))$. Unfortunately, the time required to initialize the scheme is $O(N \cdot (p/r)^2)$, so $p/r$ cannot be too large.

E. Outline of the Attack

In will be helpful in what follows to identify univariate polynomials of degree $t$ with vectors of length $t + 1$. Specifically, we will identify the degree-$t$ polynomial $p(y) = a_0 + a_1 y + \cdots + a_t y^t$ with its coefficient vector $\vec{p} = (a_0, a_1, \ldots, a_t)$.

Let $f_i(y) = F(x_i, y)$, where $F$ is the bivariate polynomial chosen by the authority and $x_i$ is the point associated with node $i$. The polynomial $s_i$ given to node $i$ is thus identified with the vector

$$\vec{s}_i = \vec{f}_i + \alpha_i \cdot \vec{g} + \beta_i \cdot \vec{h}.$$ The crucial observation underlying our attack is that the “noise” added to $\vec{f}_i$ is drawn from a low-dimensional linear subspace spanned by the two vectors $\vec{g}$ and $\vec{h}$. The attack proceeds by first identifying this “noise space”, then finding the noise polynomials $\vec{g}$ and $\vec{h}$, and finally solving for the bivariate polynomial $F$. We describe these steps in the three sections that follow.

E.1. Identifying the Noise Space

The attack begins by corrupting $n = t + 3$ nodes with associated points $x_0, \ldots, x_{t+2}$. This gives a set of $n$ vectors $\{\vec{s}_i\}_{i=0}^{t+2}$ with

$$\vec{s}_i = \vec{f}_i + \alpha_i \cdot \vec{g} + \beta_i \cdot \vec{h}.$$ The “noise space” is the vector space spanned by $\vec{g}$ and $\vec{h}$, and we now show how to identify this space. We use the fact that the $\vec{f}_i$ are all derived from the same bivariate polynomial $F$. Thus, if we write $F(x, y)$ as $F(x, y) = \sum_{j=0}^t F_j(x) y^j$ (where each $F_j$ is a univariate degree-$t$ polynomial), then for every node $i$ we have

$$\vec{f}_i = (F_0(x_i), \ldots, F_t(x_i)).$$
Recall now the Lagrange interpolation formula: If $P$ is a degree-$t$ polynomial, then for any set of $t + 1$ points $X = \{x_0, x_1, \ldots, x_t\}$ and any point $x$, it holds that

$$P(x) = \sum_{i=0}^{t} P(x_i) \prod_{j \neq i}^{t} \frac{x - x_j}{x_i - x_j} .$$

In particular, this formula applies to each of the polynomials $F_j$. If we compromise $t + 3$ nodes with points $x_0, x_1, \ldots, x_t, x_{t+1}, x_{t+2}$ and set $X = \{x_0, x_1, \ldots, x_t\}$, then we have

$$\vec{f}_{t+1} - \sum_{i=0}^{t} L(X, x_{t+1}, i) \cdot \vec{f}_i = 0$$

and

$$\vec{f}_{t+2} - \sum_{i=0}^{t} L(X, x_{t+2}, i) \cdot \vec{f}_i = 0 . \quad (1)$$

Note that we can compute explicitly all the coefficients $L(X, x, i)$ in the equations above. Taking the same linear combinations of the $\{\vec{s}_i\}$ we get

$$\vec{v} \defeq \vec{s}_{t+1} - \sum_{i=0}^{t} L(X, x_{t+1}, i) \cdot \vec{s}_i = \left( \alpha_{t+1} - \sum_{i=0}^{t} \alpha_i \cdot L(X, x_{t+1}, i) \right) \vec{g} + \left( \beta_{t+1} - \sum_{i=0}^{t} \beta_i \cdot L(X, x_{t+1}, i) \right) \vec{h} \in \text{span}(\vec{g}, \vec{h}).$$

Similarly, $\vec{v}' \defeq \vec{s}_{t+2} - \sum_{i=0}^{t} L(X, x_{t+2}, i) \cdot \vec{s}_i \in \text{span}(\vec{g}, \vec{h})$.

Since the $\alpha$’s and the $\beta$’s are chosen independently and uniformly from $[-u, u]$, it is easy to prove that $\vec{v}$ and $\vec{v}'$ span the entire space $\text{span}(\vec{g}, \vec{h})$ except with probability at most $1/2u$. Experimentally, we find that $\vec{v}$ and $\vec{v}'$ span the entire space almost surely.

G. Finding $g$ and $h$

Having computed two polynomials $v, v'$ whose associated vectors $\vec{v}, \vec{v}'$ span the noise space, we now set out to find the original polynomials $g, h$. Here we use the fact that $g, h$ are such that $g(x_i), h(x_i)$ are “small” (namely, in $[0, r]$) for all the $x_i$’s.

Consider the $n'$-dimensional integer lattice $\Lambda$ spanned by the rows of the following matrix:

$$\begin{bmatrix}
  v(x_0) & v(x_1) & \cdots & v(x_{n'-1}) \\
  v'(x_0) & v'(x_1) & \cdots & v'(x_{n'-1}) \\
  p & 0 & \cdots & 0 \\
  0 & p & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & p 
\end{bmatrix}, \quad (2)$$

where $n' \leq t$ will be fixed later. Define $g^* \defeq (g(x_0), \ldots, g(x_{n'-1}))$ and $h^* \defeq (h(x_0), \ldots, h(x_{n'-1}))$ (where the polynomial evaluation is done modulo $p$), and note that these are both short vectors (of length at most $r \cdot \sqrt{n'}$) in this lattice.

We argue in Appendix A (and verified experimentally) that when $n'$ is large enough so that $p^2 \cdot (4r/p)^{n'} < 1$, then with high probability the two shortest (independent and non-zero) vectors in the lattice $\Lambda$ are $(g^*, h^*)$ and the smaller of $g^*$ or $h^*$. This allows us to recover $g^*, h^*$ (and hence the polynomials $g$ and $h$) using lattice-basis reduction, as described next. Observe that to ensure $p^2 \cdot (4r/p)^{n'} < 1$, it is sufficient to set $n' > \left[ \frac{2 \log p}{\log p - \log 4r} \right]$, which is independent of the degree $t$. For the parameters suggested in [7] using $n' = 11$ is always enough. For this small dimension, standard lattice-reduction algorithms can exactly compute all the small vectors in the lattice, including the two shortest vectors that we need.

Denote by $\ell_1, \ell_2$ the two shortest (independent and non-zero) vectors in $\Lambda$. As we said above, with high probability one of these vectors is $\pm (g^* - h^*)$ and the other is the shorter of $g^*$ or $h^*$. In other words, with high probability the original vectors $g^*, h^*$ belong to the set $\{\ell_1, \ell_2, \pm (\ell_1 \pm \ell_2)\}$. We can identify $g^*, h^*$ using the fact that $g^*, h^* \in [0, r]^{n'}$. (In fact, $g^*, h^*$ are uniform in $[0, r]^{n'}$ since the polynomials $g, h$ are random and the $x_i$’s are chosen subject to the constraint $g(x_i), h(x_i) \in [0, r]$.) Thus, with high probability the only vectors in the set $\{\ell_1, \ell_2, \pm (\ell_1 \pm \ell_2)\}$ that belong to $[0, r]^{n'}$ are the original $g^*$ and $h^*$. So, given $\ell_1, \ell_2$ the original vectors $g^*, h^*$ can be easily found.

H. Solving for $F$

Once we have recovered $g$ and $h$, we can solve for $F$ itself. Recall that each of the $s_i$ obtained from a compromised node satisfies

$$\vec{s}_i = \vec{f}_i + \alpha_i \cdot \vec{g} + \beta_i \cdot \vec{h},$$

where $\alpha_i, \beta_i \in [-u, u]$. Using the fact that $F$ is symmetric, we have

$$s_i(x_j) = \alpha_i \cdot g(x_j) - \beta_i \cdot h(x_j) = f_i(x_j) = f_j(x_i) = s_j(x_i) - \alpha_j \cdot g(x_i) - \beta_j \cdot h(x_i) \quad (3)$$

for all $i \neq j$. Having compromised $n = t + 3$ nodes, this gives a set of $\binom{n}{2}$ linear equations in the $2n$ unknowns $\{\alpha_i, \beta_i\}_{i=1}^{n}$. Naively, we would expect this system to have full rank when $\binom{n}{2} \geq 2n$, in which case we could solve for all the $\alpha_i, \beta_i$ and then recover the $f_i$ and $F$ itself. However, this is not the case: the system is under-defined, even if we add to the system the constraints from Eq. (1). In fact, the space of solutions to this system of equations turns out to have dimension exactly three, irrespective of $t$ or $n$. (See Appendix B for an explanation.)

Since we know that $\alpha_i, \beta_i \in [-u, u]$ for all $i$, we can exhaustively search for the desired solution as follows: Set the values of three of the $\alpha$’s and $\beta$’s to values in $[-u, u]$; then solve the linear system for the rest of the $\alpha$’s and $\beta$’s and check whether they also lie in the desired range. (Heuristically, we expect that with overwhelming probability
there will be a unique solution to the system of linear equations that also satisfies $\forall i : \alpha_i, \beta_i \in [-u, u]$, and this is confirmed by our experiments.) This exhaustive search can be done in time $O(t^3 + t \cdot (2u)^3)$ by solving the system parametrically (in time $O(t^3)$) and then enumerating through $(2u)^3$ settings of the first three $\alpha$’s and $\beta$’s until the desired solution is found.

For large $u$, one could also use lattice-reduction techniques to eliminate the exhaustive search for the $\alpha$’s and $\beta$’s. This follows from the observation that the set of solutions to our linear system forms a dimension-three integer lattice, and the desired solution of $\alpha$’s and $\beta$’s is a short vector in that lattice.

I. Experimental Verification

We implemented our attack both in C++ using NTL (http://shoup.net/ntl) and in Sage [4] using Damien Stehlé’s fpLLL implementation (http://perso.ens-lyon.fr/damien.stehle) to carry out the LLL reduction. The source code of our attack (in Sage) is available online at http://www.bitbucket.org/mlal6/algebraic_attacks/noise_poly.py. Our attack ran quickly, and was successful the vast majority of the time; see Table II for representative results. Note that what prevented us from carrying out our attack on larger parameter sets was not the time required for the attack, but the time required to initialize the system!

III. The Message Authentication Schemes of Zhang et al.

Zhang, Subramanian, and Wang [6] proposed schemes for message authentication in sensor networks. They begin by describing an initial scheme, called Scheme-I in their paper, that allows a base station to authenticate a message for a set of nodes. This scheme is information-theoretically secure as long as a bounded number of messages are authenticated, and a bounded number of nodes are compromised. We describe this scheme here.

Let $p$ be a prime. The master secret key, stored by the base station, is a bivariate polynomial $F \in \mathbb{Z}_p[x, y]$ of degree $d_n$ in $x$ and degree $d_m$ in $y$. The secret key for a node $i$ is the univariate polynomial $f_i(\cdot) \overset{\Delta}{=} F(i, \cdot)$. The authentication tag for a message $m \in \mathbb{F}$ is the univariate polynomial $f_m(\cdot) \overset{\Delta}{=} F(\cdot, m)$. Node $i$ can verify the tag $f_m$ on a message $m$ by checking whether $f_m(i) \overset{\Delta}{=} f_i(m)$.

The master secret key can be recovered in its entirety if either $d_n + 1$ nodes are compromised, or if $d_m + 1$ messages are authenticated by the base station. If no nodes are compromised and at most $d_n$ messages are authenticated, or if no messages have been authenticated and at most $d_m$ nodes have been compromised, the scheme is information-theoretically secure (with probability of forgery $1/p$).

Zhang et al. present a series of extensions to this basic scheme in their paper. Scheme-II, as above, enables the base station to authenticate messages for nodes (i.e., multicast), and Scheme-IV allows authentication between nodes (i.e., many-to-many communication). Zhang et al. also propose a Scheme-III, but they themselves show that it is not secure.

A. Scheme-II and How to Break it

To enhance the security of Scheme-I, Zhang et al. suggest to add noise in the free term of the various polynomials. Specifically, fix a noise parameter $r < p/2$. The secret key of a node $i$ is now the univariate polynomial $s_i(\cdot) = F(i, \cdot) + \gamma_i$, where $\gamma_i$ is chosen uniformly in $[0, r]$. Similarly, the authentication tag for a message $m$ is now the univariate polynomial $t_m(\cdot) = F(\cdot, m) + \gamma_m$, where $\gamma_m$ is chosen uniformly in $[0, r]$. Node $i$ verifies the authentication tag $t_m$ on a message $m$ by checking whether $|s_i(m) - t_m(i)| \leq r$, where elements of $\mathbb{Z}_p$ are viewed as being in the range $[-p/2, p/2]$. Zhang et al. claim that an attack on this scheme requires complexity at least $r_{\min}(d_m, d_n) + 1$, even if an arbitrary number of nodes are compromised and an arbitrary number of messages are authenticated (cf. Theorems 3.2 and 3.3 in [6]).

This scheme is, in fact, easy to break. Noise is only introduced in the free term, so most of the coefficients of the master polynomial can be recovered by simple interpolation. If $F(x, y) = \sum_{i,j} F^j_i x^i y^j$, then by compromising $d_n + 1$ nodes an attacker can recover all the $F^j_i$’s with $j > 0$, and after seeing $d_m + 1$ authentication tags an attacker can recover all the $F^j_i$’s with $i > 0$. Compromising $d_n + 1$ nodes and seeing $d_m + 1$ authentication tags thus allows the attacker to recover all the coefficients of $F$ except for the free term. The free term can then approximated by finding any element of $\mathbb{Z}_p$ for which the resulting polynomial $F(x, y)$ gives node keys and authentication tags whose free term is close to the free term of the keys and tags already observed.

B. Scheme-IV and How to Break It

Scheme-III and Scheme-IV in [6] were designed to authenticate many-to-many communication. These schemes extend Scheme-I by using a tri-variate master polynomial whose three variables correspond to senders, receivers, and messages. Namely, the master key of the underlying scheme is a polynomial $F(x, y, z)$. A node $i$ is given two secret keys: the bivariate polynomial $F(i, \cdot, \cdot)$ (to be used when it acts as a sender), and the bivariate polynomial $F(\cdot, i, \cdot)$ (for when it acts as a receiver). The tag for a message $m$ sent by node $i$ is the univariate polynomial $F(i, \cdot, m)$; and a receiver $j$ verifies this tag in the obvious way. Scheme-III is obtained from this underlying scheme by adding noise to the free term, but Zhang et al. observe that the resulting scheme is not secure. Hence, in Scheme-IV they adopt the perturbation polynomial technique from [7] as described next.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$r$</th>
<th>$t$</th>
<th>$u$</th>
<th>setup time (minutes)</th>
<th>attack time (minutes)</th>
<th>successful attempts</th>
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<td>76</td>
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<td>60</td>
<td>10</td>
<td>7/7</td>
</tr>
<tr>
<td>$2^{36} - 5$</td>
<td>$2^{24}$</td>
<td>77</td>
<td>2</td>
<td>1060</td>
<td>8</td>
<td>2/2</td>
</tr>
</tbody>
</table>

TABLE II
SUCCESSFUL ATTACKS ON THE SCHEME OF ZHANG ET AL. TIMINGS REFLECT AN IMPLEMENTATION IN SAGE, RUNNING ON AN INTEL® XEON® CPU X7460 @ 2.66GHZ WITH 128GB RAM.
In Scheme-IV there are noise parameters $u, r$ with $u < r < p/4$. The master secret is again a tri-variate polynomial $F \in \mathbb{Z}_p[x,y,z]$ of degree $d_y$ (the “sender degree”) in $x$, degree $d_r$ (the “receiver degree”) in $y$, and degree $d_m$ (the “message degree”) in $z$. Two univariate “noise polynomials” $g(x)$ (of degree $d_y$) and $h(y)$ (of degree $d_r$) are also chosen. These define the sender ID-space $\text{Small}_S$ (resp., the receiver ID-space $\text{Small}_R$), which contains all the points on which the value of $g$ (resp., $h$) is “small”; i.e.,

$$\text{Small}_S \overset{\text{def}}{=} \{x : g(x) \in [0,r/u]\}$$

$$\text{Small}_R \overset{\text{def}}{=} \{y : h(y) \in [0,r/u]\}.$$

The scheme works as follows:

- Each node is given two keys: one for when it acts as a sender, and one for when it acts as a receiver.
- The sender secret key consists of an identity $i \in \text{Small}_S$ and the bivariate polynomial $a_i(\cdot, \cdot) = F(i,y,z) + \alpha_i \cdot h(y) + \beta_i$, with $\alpha_i$ chosen uniformly in $[0,u]$ and $\beta_i$ chosen uniformly in $[0,r/2]$.
- Similarly, the receiver secret key consists of an identity $j \in \text{Small}_R$ and the bivariate polynomial $b_j(x,z) = F(x,j,z) + \gamma_j \cdot g(x) + \delta_j$ with $\gamma_j$ chosen uniformly in $[0,u]$ and $\delta_j$ chosen uniformly in $[0,r]$.
- The authentication tag computed by sender $i$ on the message $m$ consists of the univariate polynomial $t_{i,m}(y) \overset{\text{def}}{=} a_i(y,m) + \eta_m = F(i,y,m) + \alpha_i \cdot h(y) + \beta_i + \eta_m$, where $\eta_m$ is uniform in $[0,r/2]$. Receiver $j$ verifies this tag by checking that $|b_j(i,m) - t_{i,m}(j)| \leq 2r$.
- This scheme can be broken much as in the case of Scheme-II. A key observation is that noise is only introduced in the coefficients that are independent of the message-variable $z$. Partition the master polynomial into one polynomial that depends on $z$ and another that does not:

$$F(x,y,z) = \sum_{i,j,k} F_{i,j,k} x^i y^j z^k = \sum_{k=1}^{d_y} z^k \sum_{i,j} F_{i,j,k} x^i y^j + \sum_{i,j} F_{i,j,0} x^i y^j.$$

Let $h(y) = \sum_j h_j \cdot y^j$. Then the secret key for node with sender-ID $w$ is

$$a_w(y,z) = \sum_{j=0}^{d_y} \sum_{k=1}^{d_z} \left( \sum_{i=0}^{d_x} F_{i,j,k} w^i \right) y^j z^k + \sum_{j=1}^{d_y} \left( \alpha_w h_j + \sum_{i=0}^{d_z} F_{i,j,0} w^i \right) y^j + \left( \alpha_w h_0 + \beta_w + \sum_{i=0}^{d_z} F_{i,0,0} w^i \right).$$

Let $\eta_m = \sum_j h_j \cdot y^j$. Then the message polynomial $m(x,y)$ is

$$m(x,y) = \sum_{j=1}^{d_m} \left( \sum_{i=0}^{d_x} F_{i,j,0} w^i \right) y^j.$$

Observe that for $k > 0$, the coefficient of $y^j z^k$ in $a_w$ depends only on $F1$ and not on the noise. Similarly, for $k > 0$ the coefficient of $x^i z^k$ in the receiver polynomial $b_w(x,z)$ depends only on $F1$ and not on the noise. This means that once the attacker compromises $d_y + 1$ senders or $d_r + 1$ receivers, it can fully recover the polynomial $F1$. Then, the only part of the master secret key that the attacker is missing is $F2(x,y)$, which is independent of the message variable $z$. This allows easy forgery, as described next.

Given a tag $t_{x^*,m}(y)$ computed by a non-compromised sender $x^*$ on a message $m$, the attacker (who knows $F1$) can compute the polynomial

$$\Delta(y) \overset{\text{def}}{=} t_{x^*,m}(y) - F1(x^*, y, m) = F2(x^*, y) + \alpha_{x^*} h(y) + \beta_{x^*} + \eta_m.$$

The attacker can now forge the tag of any message $m'$ as sent by the same $x^*$, by setting

$$\hat{t}_{x^*,m'}(y) \overset{\text{def}}{=} F1(x^*, y, m') + \Delta(y)$$

$$= F1(x^*, y, m') + F2(x^*, y) + \alpha_{x^*} h(y) + \beta_{x^*} + \eta_{m'}.$$

Note that this is exactly the tag that the sender $x^*$ would have sent if it chose $\eta_{m'} = \eta_{m^*}$, which means that this is a valid tag for $m'$ and would therefore be accepted by all the receivers.

Alternatively, the attacker can apply an attack similar to the one from Section II (using the fact that $g, h$ have “small values” on all the identities) to recover also the remaining master polynomial $F2(x,y)$, and thereafter it can forge messages for any sender. We omit the details.

IV. CONCLUSION

We have shown attacks on the schemes from [7], [5], [6], which are all based on “perturbation polynomials”. Our attacks show that the modified schemes are no better — and may, in fact, be worse — than the information-theoretically secure schemes they are based on. Our results cast doubt on the viability of the “perturbation polynomials” technique as an approach for designing secure cryptographic schemes.

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REFERENCES


APPENDIX

A. The Shortest Vectors in the Lattice from Eq. (2)

Here we justify the claims made in Section II-G. Recall the setting: we have an integer lattice \( \Lambda \) defined by taking integer linear combinations of the rows of the following matrix:

\[
\begin{bmatrix}
g(x_0) & g(x_1) & \cdots & g(x_n-1) \\
h(x_0) & h(x_1) & \cdots & h(x_n-1) \\
p & 0 & \cdots & 0 \\
0 & p & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & p
\end{bmatrix}
\]

(4)

(We have substituted \( g, h \) in place of \( v, v' \); since \( v, v' \) and \( g, h \) span the same space, the lattice is unchanged.) Recall that \( g^* \) (resp., \( h^* \)) denotes the first (resp., second) row of the lattice above, and that the length of \( g^*, h^* \) is at most \( r \cdot \sqrt{n'} \). Let \( \ell_1, \ell_2 \) denote the two shortest (independent and non-zero) vectors in \( \Lambda \). We provide a heuristic argument that \( \ell_1, \ell_2 \in \{ g^*, h^*, \pm (g^* - h^*) \} \) with high probability over the initial choice of \( g^*, h^* \).

The polynomials \( g \) and \( h \) are chosen during system set-up as random polynomials of degree \( t \), and the \( x_i \) are chosen such that \( g(x_i), h(x_i) \in [0, r] \). Since \( n' \leq t \), the values of \( g(x_0), \ldots, g(x_{n'-1}) \) are independent and uniform in \([0, r]\) (and similarly for \( h \)). That is, \( g^* \) and \( h^* \) are independent and uniform in \([0, r^n]\).

We note that we expect the vector \( g^* - h^* \) to be shorter than both \( g^*, h^* \) (since the expected size of each entry in \( g^* - h^* \) is roughly \( r/3 \), as compared to \( r/2 \) for each entry in \( g^*, h^* \)). We ask what is the probability that there exist some \( a, b \in \mathbb{Z}_p \) (with \( (a, b) \notin \{(0, \pm 1), (\pm 1, 0), (\pm 1, -1)\} \)) such that the vector \( ag^* + bh^* \mod p \) is shorter than both \( g^* \) and \( h^* \) (where \( \mod p \) operation maps integers into the range \([-p/2, p/2] \)). We distinguish between “small pairs” where \( |a|, |b| < p/4r \) and “large pairs” where at least one of \( |a|, |b| \) is at least \( p/4r \).

- For a “small pair”, we have no reduction mod \( p \) (since \( ag^* + bh^* \) is already in the range \([-p/2, p/2] \)). Hence, there exists a “small pair” as needed if and only if the integer lattice that is spanned by only two vectors \( g^*, h^* \) contains a vector other than \( \pm (g^* - h^*) \) which is shorter than both \( g^*, h^* \). When \( (a, b) \notin \{(0, \pm 1), (\pm 1, 0), (\pm 1, -1)\} \), the expected size of each entry in \( ag^* + bh^* \mod p \) is larger than \( r/2 \) and so we expect \( ag^* + bh^* \) to be longer than \( g^*, h^* \). Hence, we expect the two shortest vectors in the lattice spanned by \( g^*, h^* \) to be in the set \( \{ \pm g^*, \pm h^*, \pm (g^* - h^*) \} \) except with probability exponentially small in \( n' \).

- For any fixed “large pair” \((a, b)\), the distribution of the vector \( ag^* + bh^* \mod p \) is rather close to the uniform distribution over \([-p/2, p/2] \). To see this, assume \(|a| > p/4r \). Fix \( b \) and \( h^* \) to some arbitrary values, and consider the residual distribution on \( ag^* + bh^* \mod p \) induced by choosing \( g^* \in [0, r]^n \). Consider a “simplified setting” where the entries of \( g^* \) are chosen from the real interval \([0, r]\) (instead of only the integers in this interval). In this setting, each entry of \( ag^* + bh^* \mod p \) would be chosen from a distribution that has statistical distance at most 3/4 from the uniform distribution on \([-p/2, p/2] \). When \(|a| \) is larger still, the distribution gets even closer to uniform. For example, for \(|a| = \Theta(p) \) the distance from uniform is \( O(1/r) \). The quantization to integers of course changes the distribution, but does not change substantially the probability that the resulting vector is short.

Making the heuristic assumption that the length of \( ag^* + bh^* \mod p \) is distributed as if that vector were uniform in \([-p/2, p/2]\), we can estimate the probability that this vector lies in the ball of radius \( r\sqrt{n'} \) around the origin. The volume of such a ball is \( \frac{(r\sqrt{n'})^n \pi^{n'/2}}{(n'!/2)!} \approx r^n (2\pi e)^{n'/2} \approx (4r)^n \).

Hence, the probability that a uniformly distributed vector in \([-p/2, p/2]^n\) has length below \( r\sqrt{n'} \) is upper-bounded by \((4r)^n\), and we can heuristically use the same bound also for the length of \( ag^* + bh^* \mod p \) for any fixed “large pair” \((a, b)\). As there are fewer than \( p^2 \) “large pairs”, a union bound implies that when \( p^2 \cdot (4r)^n \ll 1 \) we expect to have \(|ag^* + bh^* \mod p| \geq r\sqrt{n'} \) for every “large pair”.

Experimentally, we observe that for even moderate values of \( n' \), the two smallest vectors in the lattice are indeed \( \pm (g^* - h^*) \) and the smaller of \( g^*, h^* \). Specifically, we ran the following experiment: generate \( g^*, h^* \) uniformly in \([0, r]^n\) and then run LLL on the lattice from Eq. (4) to compute the shortest vectors \( \ell_1, \ell_2 \) of the resulting lattice. Call it a “success” if \( g^*, h^* \in \{ \ell_1, \ell_2, \ell_1 \pm \ell_2 \} \). For each setting of \( p \) and \( r \), we then determined the minimum value of \( n' \) for which a success occurred at least 95% of the time (in 200 trials). The results are in Table III.
B. Solving for the α’s and β’s

Here we explain why the linear system of equations described in Section II-H is under-defined, and why the vector space of solutions has dimension 3.

Every solution to our system of linear equations must correspond to a bivariate degree-$t$ polynomial $F$ (due to the inclusion of Eq. (1)) which is symmetric (due to the equations from Eq. (3)). Moreover, for each node associated with the point $x_i$ the polynomial $F$ induces coefficients $\bar{f}_i$ such that $\bar{s}_i - \bar{f}_i$ belongs to the vector space spanned by $\bar{g}$ and $\bar{h}$. Let $F, F'$ be two polynomials satisfying these constraints, and consider their difference polynomial $D = F - F'$. This polynomial $D$ satisfies the following three conditions:

- $D$ is a bivariate degree-$t$ polynomial (since $F$ and $F'$ are);
- $D$ is symmetric (since $F$ and $F'$ are);
- For every $i$, if we let $\bar{d}_i$ denote the coefficients of the univariate polynomial $D(x_i, \cdot)$, then all the $\bar{d}_i$’s belong to the vector space spanned by $\bar{g}$ and $\bar{h}$.

We now show that there are exactly three degrees of freedom in choosing a polynomial $D$ with these properties. Denote the matrix of coefficients of $D$ by $[D]$, and denote by $[d]$ the matrix whose $i$th row is the vector $\bar{d}_i$ for $i = 0, 1, \ldots, t$. Then $[d] = V \cdot [D]$, where $V$ is a Vandermonde matrix:

$$
\begin{pmatrix}
    d_{0,0} & d_{0,1} & \cdots & d_{0,t} \\
    d_{1,0} & d_{1,1} & \cdots & d_{1,t} \\
    \vdots & \vdots & \ddots & \vdots \\
    d_{t,0} & d_{t,1} & \cdots & d_{t,t}
\end{pmatrix}
\begin{pmatrix}
    x_0 & x_1 & \cdots & x_t
\end{pmatrix}
= 
\begin{pmatrix}
    1 & x_0 & \cdots & x_0^t \\
    1 & x_1 & \cdots & x_1^t \\
    \vdots & \vdots & \ddots & \vdots \\
    1 & x_t & \cdots & x_t^t
\end{pmatrix}
\begin{pmatrix}
    D_{0,0} & D_{0,1} & \cdots & D_{0,t} \\
    D_{1,0} & D_{1,1} & \cdots & D_{1,t} \\
    \vdots & \vdots & \ddots & \vdots \\
    D_{t,0} & D_{t,1} & \cdots & D_{t,t}
\end{pmatrix}

$$

The conditions on the polynomial $D$ translate to the conditions that $[D]$ is a $(t + 1) \times (t + 1)$ symmetric matrix, and that the rows of $[d]$ are in the vector space spanned by $\bar{g}$ and $\bar{h}$. The last condition can be expressed in matrix notation by saying that there exists a $(t + 1) \times 2$ matrix $X$ such that

$$
[d] = X \cdot \begin{pmatrix} \bar{g} \\ \bar{h} \end{pmatrix}.
$$

To obtain a $D$ satisfying these conditions, choose an arbitrary symmetric $2 \times 2$ matrix $R$ and set

$$
[D] := \begin{pmatrix} \bar{g}^T \\ \bar{h}^T \end{pmatrix} \cdot R \cdot \begin{pmatrix} \bar{g} \\ \bar{h} \end{pmatrix},
$$

where $\bar{g}^T$ and $\bar{h}^T$ (the transpose of $\bar{g}$ and $\bar{h}$, respectively) are column vectors. This ensures that $[D]$ is a $(t + 1) \times (t + 1)$ symmetric matrix, and moreover

$$
[d] = V \cdot [D] = V \cdot \begin{pmatrix} \bar{g}^T \\ \bar{h}^T \end{pmatrix} \cdot R \cdot \begin{pmatrix} \bar{g} \\ \bar{h} \end{pmatrix} \cdot X
$$

as needed. Since there are three degrees of freedom in choosing a symmetric $2 \times 2$ symmetric matrix, we get exactly three degrees of freedom for $D$. 

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<tr>
<td>2^4</td>
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<td>10</td>
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</table>

TABLE III

DIMENSION $n'$ NEEDED FOR RECOVERY OF $g^*, h^*$. 

<table>
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<tr>
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<th>$r$</th>
<th>$n'$</th>
</tr>
</thead>
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